Solution to HW7

Leon Li

ylli@math.cuhk.edu.hk

MATH 2020 B HW 7

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Thomas' Calculus (12th Ed.)

§16.3: 7, 14, 19, 20, 30, 38

§16.3

Finding Potential Functions

In Exercises 7–12, find a potential function f for the field $\mathbf{F}$.

7. $\mathbf{F} = 2x\mathbf{i} + 3y\mathbf{j} + 4z\mathbf{k}$

Sol) $\nabla f(x,y,z) = \vec{F}(x,y,z) \Rightarrow \frac{\partial f}{\partial x} = 2x \Rightarrow f(x,y,z) = x^2 + g(y,z).$

$\therefore \nabla(x^2 + g(y,z)) = \vec{F}(x,y,z) \Rightarrow \frac{\partial g}{\partial y} = 3y \Rightarrow g(y,z) = \frac{3y^2}{2} + h(z).$

$\therefore \nabla(x^2 + \frac{3y^2}{2} + h(z)) = \vec{F}(x,y,z) \Rightarrow h'(z) = 4z \Rightarrow h(z) = 2z^2 + C$

$\therefore f(x,y,z) = x^2 + \frac{3y^2}{2} + 2z^2 + C$ //

Exact Differential Forms

In Exercises 13–17, show that the differential forms in the integrals are exact. Then evaluate the integrals.

14. $\displaystyle\int_{(1,1,2)}^{(3,5,0)} yz\,dx + xz\,dy + xy\,dz$

Sol) Let $\vec{F}(x,y,z) = yz\,\vec{i} + xz\,\vec{j} + xy\,\vec{k} = M(x,y,z)\vec{i} + N(x,y,z)\vec{j} + P(x,y,z)\vec{k}$.

then $yz\,dx + xz\,dy + xy\,dz$ is exact $\Leftrightarrow$ $\begin{cases} M_y = N_x & - ① \\ M_z = P_x & - ② \\ N_z = P_y & - ③ \end{cases}$

Checking ①–③:

①: $LHS = M_y = z = N_x = RHS$

②: $LHS = M_z = y = P_x = RHS$

③: $LHS = N_z = x = P_y = RHS$

$\therefore$ $yz\,dx + xz\,dy + xy\,dz$ is exact.

$\therefore$ There exists $f: \mathbb{R}^3 \to \mathbb{R}$ such that $\nabla f(x,y,z) = \vec{F}(x,y,z)$.

$\nabla f(x,y,z) = \vec{F}(x,y,z) \Rightarrow \dfrac{\partial f}{\partial x} = yz \Rightarrow f(x,y,z) = xyz + g(y,z)$.

$\therefore \nabla(xyz + g(y,z)) = \vec{F}(x,y,z) \Rightarrow xz + \dfrac{\partial g}{\partial y} = xz \Rightarrow g(y,z) = C$

$\therefore f(x,y,z) = xyz + C$.

$\therefore \displaystyle\int_{(1,1,2)}^{(3,5,0)} yz\,dx + xz\,dy + xy\,dz = f(3,5,0) - f(1,1,2) = C - (2+C) = -2$ //

Finding Potential Functions to Evaluate Line Integrals

Although they are not defined on all of space R^3, the fields associated with Exercises 18–22 are simply connected and the Component Test can be used to show they are conservative. Find a potential function for each field and evaluate the integrals as in Example 6.

19. $\displaystyle\int_{(1,1,1)}^{(1,2,3)} 3x^2\,dx + \frac{z^2}{y}\,dy + 2z\ln y\,dz$

20. $\displaystyle\int_{(1,2,1)}^{(2,1,1)} (2x\ln y - yz)\,dx + \left(\frac{x^2}{y} - xz\right)dy - xy\,dz$

EXAMPLE 6 Show that $y\,dx + x\,dy + 4\,dz$ is exact and evaluate the integral

$$\int_{(1,1,1)}^{(2,3,-1)} y\,dx + x\,dy + 4\,dz$$

over any path from $(1, 1, 1)$ to $(2, 3, -1)$.

Sol) (19) Let $\vec{F}(x,y,z) = 3x^2\vec{i} + \frac{z^2}{y}\vec{j} + 2z\ln y\,\vec{k} = M(x,y,z)\vec{i} + N(x,y,z)\vec{j} + P(x,y,z)\vec{k}$

then $3x^2dx + \frac{z^2}{y}dy + 2z\ln y\,dz$ is exact $\Longleftrightarrow \begin{cases} M_y = N_x & - \ (1) \\ M_z = P_x & - \ (2) \\ N_z = P_y & - \ (3) \end{cases}$

Checking (1) - (3): (1): $M_y = 0 = N_x$.

(2): $M_z = 0 = P_x$. (3): $N_z = \frac{2z}{y} = P_y$.

$\therefore$ $3x^2dx + \frac{z^2}{y}dy + 2z\ln y\,dz$ is exact.

$\therefore$ There exists $f : \{(x,y,z)\in\mathbb{R}^3 \mid y>0\} \to \mathbb{R}$ such that $\nabla f(x,y,z) = \vec{F}(x,y,z)$.

$\nabla f(x,y,z) = \vec{F}(x,y,z) \Rightarrow \frac{\partial f}{\partial x} = 3x^2 \Rightarrow f(x,y,z) = x^3 + g(y,z)$.

$\therefore$ $\nabla(x^3 + g(y,z)) = \vec{F}(x,y,z) \Longrightarrow \frac{\partial g}{\partial y} = \frac{z^2}{y} \Rightarrow g(y,z) = z^2\ln y + h(z)$

$\therefore$ $\nabla(x^3 + z^2\ln y + h(z)) = \vec{F}(x,y,z) \Rightarrow 2z\ln y + h'(z) = 2z\ln y \Rightarrow h(z) = C$

$\therefore$ $f(x,y,z) = x^3 + z^2\ln y + C$.

$\therefore$ $\displaystyle\int_{(1,1,1)}^{(1,2,3)} 3x^2dx + \frac{z^2}{y}dy + 2z\ln y\,dz = f(1,2,3) - f(1,1,1) = (1 + 9\ln 2 + C) - (1 + C) = 9\ln 2$ //

(20) Let $\vec{F}(x,y,z)=(2x\ln y - yz)\vec{i}+(\frac{x^2}{y}-xz)\vec{j}-xy\vec{k} = M(x,y,z)\vec{i}+N(x,y,z)\vec{j}+P(x,y,z)\vec{k}$

then $(2x\ln y-yz)dx+(\frac{x^2}{y}-xz)dy-xydz$ is exact $\Leftrightarrow$ $\begin{cases} M_y = N_x & - \ ① \\ M_z = P_x & - \ ② \\ N_z = P_y & - \ ③ \end{cases}$

Checking ①-③: ①: $M_y = \frac{2x}{y} - z = N_x$.

②: $M_z = -y = P_x$. ③: $N_z = -x = P_y$.

$\therefore (2x\ln y-yz)dx+(\frac{x^2}{y}-xz)dy-xydz$ is exact.

$\therefore$ There exists $f:\{(x,y,z)\in\mathbb{R}^3 \mid y>0\}\to\mathbb{R}$ such that $\nabla f(x,y,z)=\vec{F}(x,y,z)$.

$$\nabla f(x,y,z)=\vec{F}(x,y,z) \Rightarrow \frac{\partial f}{\partial x} = 2x\ln y - yz \Rightarrow f(x,y,z) = x^2\ln y - xyz + g(y,z).$$

$\therefore \nabla(x^2\ln y - xyz + g(y,z)) = \vec{F}(x,y,z) \Rightarrow \frac{x^2}{y} - xz + \frac{\partial g}{\partial y} = \frac{x^2}{y} - xz \Rightarrow g(y,z) = C.$

$\therefore f(x,y,z) = x^2\ln y - xyz + C.$

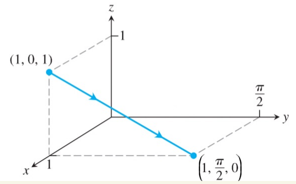
$\therefore \int_{(1,2,1)}^{(2,1,1)} (2x\ln y-yz)dx+(\frac{x^2}{y}-xz)dy-xydz = f(2,1,1)-f(1,2,1)$

$= (-2+C)-(\ln 2 - 2 + C) = -\ln 2$ //

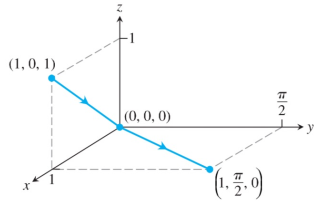
Applications and Examples

30. Work along different paths Find the work done by $\mathbf{F} = e^{yz}\mathbf{i} + (xze^{yz} + z\cos y)\mathbf{j} + (xye^{yz} + \sin y)\mathbf{k}$ over the following paths from $(1, 0, 1)$ to $(1, \pi/2, 0)$.

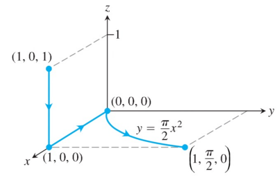
a. The line segment $x = 1, y = \pi t/2, z = 1 - t, 0 \le t \le 1$



b. The line segment from $(1, 0, 1)$ to the origin followed by the line segment from the origin to $(1, \pi/2, 0)$



c. The line segment from $(1, 0, 1)$ to $(1, 0, 0)$, followed by the x-axis from $(1, 0, 0)$ to the origin, followed by the parabola $y = \pi x^2/2, z = 0$ from there to $(1, \pi/2, 0)$



Sol) We first show that $\vec{F}(x,y,z) = M(x,y,z)\vec{i} + N(x,y,z)\vec{j} + P(x,y,z)\vec{k}$ is conservative.

i.e. $\begin{cases} M_y = N_x & - ① \\ M_z = P_x & - ② \\ N_z = P_y & - ③ \end{cases}$ Checking ①-③: ①: $M_y = ze^{yz} = N_x$.

②: $M_z = ye^{yz} = P_x$. ③: $N_z = xe^{yz} + xyze^{yz} + \cos y = P_y$.

$\therefore \vec{F}(x,y,z)$ is conservative. $\therefore$ There exists $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\nabla f(x,y,z) = \vec{F}(x,y,z)$.

$$\nabla f(x,y,z) = \vec{F}(x,y,z) \Rightarrow \frac{\partial f}{\partial x} = e^{yz} \Rightarrow f(x,y,z) = xe^{yz} + g(y,z).$$

$$\therefore \nabla(xe^{yz} + g(y,z)) = \vec{F}(x,y,z) \Rightarrow xze^{yz} + \frac{\partial g}{\partial y} = xze^{yz} + z\cos y \Rightarrow g(y,z) = z\sin y + h(z).$$

$$\therefore \nabla(xe^{yz} + z\sin y + h(z)) = \vec{F}(x,y,z) \Rightarrow xye^{yz} + \sin y + h'(z) = xye^{yz} + \sin y \Rightarrow h(z) = C.$$

$$\therefore f(x,y,z) = xe^{yz} + z\sin y + C$$

(a) Work $= \int_{(1,0,1)}^{(1,\frac{\pi}{2},0)} \left(e^{yz}dx + (xze^{yz} + z\cos y)dy + (xye^{yz} + \sin y)dz\right)$

$$= f(1,\tfrac{\pi}{2},0) - f(1,0,1) = (1+C) - (1+C) = 0$$

(b) Work $= f(1,\frac{\pi}{2},0) - f(1,0,1) = 0$

(c) Work $= f(1,\frac{\pi}{2},0) - f(1,0,1) = 0$

38. Gravitational field

a. Find a potential function for the gravitational field

$$\mathbf{F} = -GmM\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}}$$

(G, m, and M are constants).

b. Let P_1 and P_2 be points at distance s_1 and s_2 from the origin. Show that the work done by the gravitational field in part (a) in moving a particle from P_1 to P_2 is

$$GmM\left(\frac{1}{s_2} - \frac{1}{s_1}\right).$$

Sol) (a) Let $-GmM = A$, so $F(x,y,z) = \frac{A}{(x^2+y^2+z^2)^{3/2}}(x\vec{i} + y\vec{j} + z\vec{k})$

We first show that $F(x,y,z) = M(x,y,z)\vec{i} + N(x,y,z)\vec{j} + P(x,y,z)\vec{k}$ is conservative.

i.e. $\begin{cases} M_y = N_x & - ① \\ M_z = P_x & - ② \\ N_z = P_y & - ③ \end{cases}$ Checking ① - ③: ①: $M_y = \frac{-3Axy}{(x^2+y^2+z^2)^{5/2}} = N_x$.

②: $M_z = \frac{-3Axz}{(x^2+y^2+z^2)^{5/2}} = P_x$. ③: $N_z = \frac{-3Ayz}{(x^2+y^2+z^2)^{5/2}} = P_y$.

$\therefore \vec{F}(x,y,z)$ is conservative. $\therefore$ There exists $f: \mathbb{R}^3\setminus\{0\} \to \mathbb{R}$ such that $\nabla f(x,y,z) = \vec{F}(x,y,z)$.

$\nabla f(x,y,z) = \vec{F}(x,y,z) \Rightarrow \frac{\partial f}{\partial x} = \frac{Ax}{(x^2+y^2+z^2)^{3/2}} \Rightarrow f(x,y,z) = \frac{-A}{(x^2+y^2+z^2)^{1/2}} + g(y,z)$.

$\therefore \nabla\left(\frac{-A}{(x^2+y^2+z^2)^{1/2}} + g(y,z)\right) = \vec{F}(x,y,z) \Rightarrow \frac{Ay}{(x^2+y^2+z^2)^{3/2}} + \frac{\partial g}{\partial y} = \frac{Ay}{(x^2+y^2+z^2)^{3/2}} \Rightarrow g(y,z) = C$.

$\therefore f(x,y,z) = \frac{-A}{(x^2+y^2+z^2)^{1/2}} + C$ //

(since s_i = distance from P_i to the origin, $i=1,2$)

(b) Work $= \int_{P_1}^{P_2} \frac{A}{(x^2+y^2+z^2)^{3/2}}(x\,dx + y\,dy + z\,dz) = f(P_2) - f(P_1) = \left(-\frac{A}{s_2} + C\right) - \left(-\frac{A}{s_1} + C\right)$

$= -A\left(\frac{1}{s_2} - \frac{1}{s_1}\right) = GMm\left(\frac{1}{s_2} - \frac{1}{s_1}\right)$ //